Air Resistance: Background

June 7, 2005

All right, now that you've played with the applet here is a sheet that

1. helps clear up whatever confusion still lingers, and
2. lets you know what the lab intended for you to learn.

Having said that, here are a few conventions: the coordinates are fixed so that $\tilde{\mathbf{y}}$ is up and $\tilde{\mathbf{x}}$ is to the right. The applet runs the equations of motions for an object with some fixed size and shape, but variable mass (m). You were allowed to hurl it with whatever initial speed you wanted (v_0), at some angle (θ) between $0°$ and $90°$. You were also allowed to toggle air resistance on or off. When you fiddled with the mass of the object with the air resistance off nothing changed, right? Let's figure out why this might be. Remember that near the earth, gravity is a force (field) that causes *constant acceleration*. In our coordinates, $a_{\mathbf{x}} = 0$ and $a_{\mathbf{y}} = -g$. With this in mind, Newton's law tells us immediately why mass doesn't affect motion of an object due to gravity alone. Observe:

$$\begin{aligned} \sum \tilde{\mathbf{F}} &= m\tilde{\mathbf{a}} \\ &= m\tilde{\mathbf{g}}. \end{aligned}$$

In other words, $\tilde{\mathbf{a}} = \tilde{\mathbf{g}}$. This result is good, since you probably don't recall seeing m in any of the constant acceleration equations of motion...

Calculus Interlude

Let's sneak in a little calculus :) To get position from acceleration we need to integrate both sides of the equation twice; once turns acceleration into velocity, once more turns velocity to position. First, let's do the components along $\mathbf{x}$:

$$a_{\mathbf{x}} = 0$$

$$\int a_{\mathbf{x}}(t)\, dt = v_{\mathbf{x}}(t) \quad (1)$$

$$\int 0\, dt = C \quad (2)$$

where C is a constant. Now,

$$\int v_{\mathbf{x}}(t)\, dt = p_{\mathbf{x}}(t) \quad (3)$$

$$\int C\, dt = Ct + D \quad (4)$$

where D is a constant and $p_{\mathbf{x}}(t)$ is the **p**osition of the object along $\mathbf{x}$ at the time t. Integrals generate these pesky undetermined constants. Not to fear though! We have all of the info we need to figure out what they are. To be more specific, we know everything about our projectile at time $t = 0$. The $\mathbf{x}$-velocity, $v_{\mathbf{x}}(0)$ is $v_0 \cos\theta$. This means that $C = v_0 \cos\theta$ in equation 2. Next, we know that $p_{\mathbf{x}}(0) = x_0$. This means that $D = x_0$ in equation 4. Written all together we have the familiar equation

$$\begin{aligned} p_{\mathbf{x}}(t) &= Ct + D \\ &= (v_0 \cos\theta)t + x_0. \end{aligned}$$

Now let's do the $\mathbf{y}$ components. Here we have:

$$a_{\mathbf{y}} = -g$$

$$\int a_{\mathbf{y}}(t)\, dt = v_{\mathbf{y}}(t) \quad (5)$$

$$\int -g\, dt = -gt + A \quad (6)$$

where A is a constant. Moving right along,

$$\begin{aligned}\int v_{\mathbf{y}}(t)\ dt &= p_{\mathbf{y}}(t) && (7)\\ \int (-gt + A)\ d &= \int -gt\ d + \int A\ d && (8)\\ &= -\frac{1}{2}gt^2 + At + B\end{aligned}$$

where B is a constant and $p_{\mathbf{y}}(t)$ is the **p**osition of the object along $\mathbf{y}$ at the time t. Just like with the $\mathbf{x}$ components, we know that $v_{\mathbf{y}}(0) = v_0 \sin\theta$ and $p_{\mathbf{y}}(0) = y_0$. Together these mean that $A = v_0 \sin\theta$ in equation 6 and $B = y_0$ in equation 9. Thankfully, this gives us

$$\begin{aligned}p_{\mathbf{y}}(t) &= -\frac{1}{2}gt^2 + At + B\\ &= -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0\end{aligned}$$

and we no longer have to take any book's word for it! These are (not surprisingly) referred to the *equations of motion* for the projectile and we obtained them from Newton's law.

Air Resistance

The applet lab was asking a lot of you when it tried to have you figure out what the force from air resistance 'looked' like with only your own experience and the applet to refer to. For reference, figure 1 shows some results from the applet. As it turns out, the drag force (resistance to the motion of a body through some *fluid*, like *air*) depends on several things. Here are some steps to help figure out just what those things might be.

First off, let's do some experiments (or at least do them in our head). When we do this, the idea is to try and change *only one thing at a time* so that we can sort out the effect of many different factors. Let's start with speed. This seems like a good place to begin since experience says you wouldn't feel any drag force when sitting on a motorcycle in the driveway, but certainly would driving on the highway.

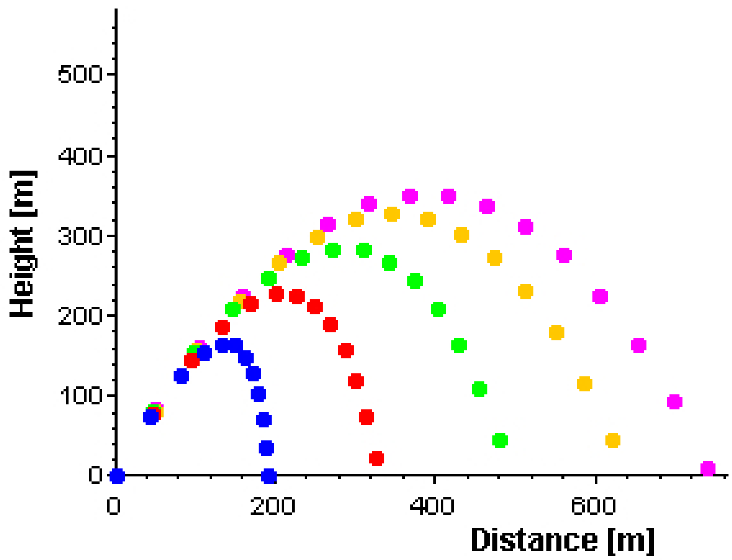


Figure 1: Results from the projectile motion applet for a fixed angle and initial velocity. Can you deduce what was changed (and how) between the curves?

As you might have guessed, the drag force (lets call it $\tilde{\mathbf{f}}^d$) must depend on the *velocity* of the projectile. When we say *velocity* we really mean the full vector *velocity* and not just speed. Think of your flat hand hanging out the window of a car moving 30 mph forward (We don't actually suggest doing this). If you didn't hold your hand still it would get pushed backwards right? If the car was moving at 80 mph it would get pushed even harder, right? *(Aside: you'd know this because, your muscles would have to work harder to provide enough force to keep your hand still, just like Newton's Third Law states).* Similarly, if the car was moving in reverse at 80 mph (not recommended!), the direction of the force on your hand would change. Now since a change in velocity immediately results in a change in the drag force, it is the *instantaneous velocity* that we are talking about here.

What if we change the fluid? As another example, think of moving your hand with palm open under water: the same principles apply. Although, now there is a difference. Is it harder or easier to move your hand under water vs. waving it through the air? The answer is most definitely harder (meaning it takes more force to move my hand underwater than at the same speed as through the air). This might make you conclude that the *density* of the fluid has something to do with the drag force, recalling that water is more dense than air.

What else haven't we tried changing? Well, one thing is the shape of the object. Back to you in the car with your arm sticking out: Is the force greater on your closed fist or your open hand at a given velocity? Any difference here suggests that the *frontal area* of the object (how 'big' it is looking down the direction of its motion) affects the drag force.

Now what about mass? In the applet, changing the mass certainly makes a difference in the motion of the projectile with air resistance on. But does the drag force depend on the mass? We know that the force of gravity does... To clarify, let's look at Newton's law:

$$\begin{aligned} \sum \tilde{\mathbf{F}} &= m\tilde{\mathbf{a}} \\ &= m\tilde{\mathbf{g}} + \tilde{\mathbf{f}}^d. \end{aligned}$$

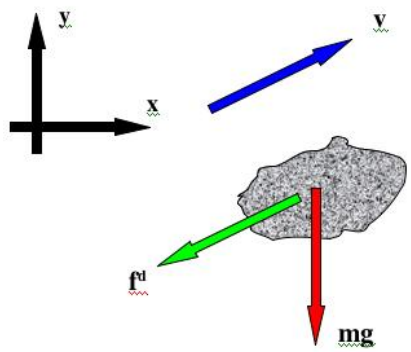
We've deduced that $\tilde{\mathbf{f}}^d$ is proportional to at least $\tilde{\mathbf{v}}$, the fluid density, ρ and the surface area, A_f. That means it might look something like $\tilde{\mathbf{f}}^d = -\rho A_f \tilde{\mathbf{v}}$. If we check quickly, this force is in the right direction (opposite $\tilde{\mathbf{v}}$) and gets bigger as each $\tilde{\mathbf{v}}$, ρ and A_f get bigger. There's one minor problem though. We don't know exactly how big $\tilde{\mathbf{f}}^d$ will be in magnitude for values of ρ, A_f and $\tilde{\mathbf{v}}$. Luckily, there's a quick fix: we let the whole expression be scaled (i.e. multiplied by) another unknown positive constant. Let's call that constant k. Why positive, you ask? Well, if we let k be negative, then that would change the direction of the drag force...and we don't want to do that. Now our *model* for the drag force looks like

$$\tilde{\mathbf{f}}^d = -k\rho A_f \tilde{\mathbf{v}}.$$

and

$$\begin{aligned} \sum \tilde{\mathbf{F}} &= m\tilde{\mathbf{a}} \\ &= m\tilde{\mathbf{g}} - k\rho A_f \tilde{\mathbf{v}}. \end{aligned}$$

The free body diagram looks something like:



Now, what does the acceleration of the projectile look like? We have

$$
\begin{aligned}
m\tilde{\mathbf{a}} &= m\tilde{\mathbf{g}} - k\rho A_f \tilde{\mathbf{v}} \\
\tilde{\mathbf{a}} &= \tilde{\mathbf{g}} - \frac{k}{m}\rho A_f \tilde{\mathbf{v}}.
\end{aligned} \tag{9}
$$

Now if the drag force was proportional to the mass as well, then it would disappear from the equation for the acceleration (check by sticking m in the numerator of the drag part of 9)! Well, this is *not* what we see in the applet (or real life for that matter). The conclusion is that the drag force doesn't seem to depend on mass...at least it's definitely *not proportional* to the mass of the object. As it turns out, we can't say any more than this just from playing with the applet (tricky, but not a trick!). So how can we be sure that drag doesn't depend on the projectile's mass at all? Here's another thought experiment for you: If you have a ping pong ball and a solid ball of the same size, would the drag force be the same on them? The key here is to do an experiment that isolates the drag force from the motion.

From what we've discussed here at least, we're left with a drag force that is proportional to the velocity and frontal area of the projectile, and the density of the fluid it's moving through. The velocity is the only thing we can really change in the applet that affects the drag force, however. Remember that in the applet we are watching the motion of the projectile, which includes the effects of both the drag and gravitational forces together. Changing the mass is changing the motion, but *not* the drag force.

And that's all for now...